Proof of Factorization Using Background Field Method of QCD

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Abstract

Factorization theorem plays the central role at high energy colliders to study standard model and beyond standard model physics. The proof of factorization theorem is given by Collins, Soper and Sterman to all orders in perturbation theory by using diagrammatic approach. One might wonder if one can obtain the proof of factorization theorem through symmetry considerations at the lagrangian level. In this paper we provide such a proof.

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Factorization theorem plays the central role at high energy colliders to study standard model and/or beyond standard model physics. The proof of the factorization theorem is very non-trivial and is given by Collins, Soper and Sterman by using diagrammatic approach to all orders in perturbation theory [1, 2] (see also [3, 4]). Factorization refers to separation of short-distance from long-distance effects in field theory. Using the factorization theorem the hadron production cross section at high energy hadronic colliders can be calculated by using the well known formula

$$\sigma(AB \to H+X) \;=\; \sum_{a,b,c,d} \int dx_1 \int dx_2 \int dz\; f_{a/A}(x_1,Q^2)\; f_{b/B}(x_2,Q^2)\; \hat{\sigma}(ab\to cd)\; D_{H/c}(z,Q^2) \tag{1}$$

where

$$f_{q/P}(x,k_T^2) = \frac{1}{2}\int dy^- \frac{d^{d-2}y_T}{(2\pi)^{d-1}} e^{ixP^+y^- + ik_T\cdot y_T} \\ \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}}\; \frac{1}{3}\mathrm{tr}_{\mathrm{color}}[\gamma^+ < P|\bar{\psi}(y^-,y_T)\;\Phi[y^-,y_T]\;\Phi^{-1}[0]\;\psi(0)|P>]. \tag{2}$$

is the parton distribution function and

$$D_{H/q}(z,P_T^2) = \frac{1}{2z}\int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}} e^{ik^+x^- + iP_T\cdot x_T/z} \\ \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}}\; \frac{1}{3}\mathrm{tr}_{\mathrm{color}}[\gamma^+ < 0|\bar{\psi}(x^-,x_T)\;\Phi[x^-,x_T]\; a_H^\dagger(P^+,0_T)a_H(P^+,0_T)\;\Phi^{-1}[0]\;\psi(0)|0>]. \tag{3}$$

is the fragmentation function [2]. In the above expression $a_H^\dagger$ is the creation operator of a hadron. The Wilson line is given by

$$\Phi[x^\mu] = \mathcal{P}\;\exp[ig\int_{-\infty}^{0} d\lambda\; h\cdot \mathcal{A}^a(x^\mu + h^\mu\lambda)\;T^a] \tag{4}$$

where h^μ is a x^μ independent four vector.

One might wonder if one can obtain the proof of factorization theorem through symmetry considerations at the lagrangian level. Such an attempt was made in [5] to prove factorization of soft and collinear divergences by using background field method of QCD [6, 7]. However, the proof presented in [5] is incomplete for QCD although it is complete for QED [8]. The main difficulty is due to the gauge fixing term in the background field method of QCD which, unlike QED, depends on the background field [6, 7]. Recently we have derived a gauge fixing identity [9] which relates the generating functional in QCD with the generating functional

in the background field method of QCD in pure gauge. In this paper we will provide a proof of the factorization theorem in high energy QCD by using this gauge fixing identity.

The most crucial statement of factorization theorem of Collins, Soper and Sterman is the appearance of the Wilson line eq. (4) in the definition of the structure functions and fragmentation functions in eqs. (2) and (3) which makes them gauge invariant. This Wilson line is responsible for cancelation of soft and collinear divergences which arise due to presence of loops and/or higher order Feynman diagrams. This Wilson line can be thought of as a quark or gluon jet propagating in a soft and/or collinear gluon cloud. One of the ideas to use background field method of QCD is to first show that this soft and collinear gluon cloud can be represented as a classical background field $\mathcal{A}_\mu^a(x)$ in an abelian-like pure gauge

$$\mathcal{A}_\mu^a = \partial_\mu \omega^a(x). \tag{5}$$

This can be shown as follows.

The Wilson line eq. (4) can be written as

$$\Phi[x^\mu] = \mathcal{P}\ \exp[ig \int_{-\infty}^{0} d\lambda\ h \cdot \mathcal{A}^a(x^\mu + h^\mu \lambda)\ T^a]$$
$$= \mathcal{P}\ \exp[ig \int_{-\infty}^{0} d\lambda\ h \cdot e^{\lambda h \cdot \partial} \mathcal{A}^a(x^\mu)\ T^a] = \mathcal{P}\ \exp[ig \frac{1}{h \cdot \partial} h \cdot \mathcal{A}^a(x^\mu)\ T^a]. \tag{6}$$

Using the Fourier transformation

$$\mathcal{A}_\mu^a(x) = \int \frac{d^4k}{(2\pi)^4} \mathcal{A}_\mu^a(k) e^{ik\cdot x} \tag{7}$$

we find from eq. (6) the phase factor

$$V = g\ \omega(k) = ig\ \frac{h \cdot \mathcal{A}(k)}{h \cdot k} \tag{8}$$

which is precisely the Eikonal Feynman vertex for soft and collinear divergences. For example if we choose $h^\mu = n^\mu$, where n^μ is a fixed lightlike vector having only "+" or "-" component [1]

$$n^\mu = (n^+, n^-, n_T) = (1, 0, 0) \qquad \text{or} \qquad n^\mu = (n^+, n^-, n_T) = (0, 1, 0) \tag{9}$$

we reproduce the divergences due to soft gluons. Similarly, if we choose $h^\mu = n_B^\mu$, where n_B^μ is a non-light like vector

$$n_B^\mu = (n_B^+, n_B^-, 0), \tag{10}$$

we reproduce the Feynman rules for the collinear divergences [1]. Multiplying a x^μ independent free vector h^μ and dividing $h\cdot\partial$ from left in eq. (5) we find

$$\omega^a(x) = \frac{1}{h\cdot\partial} h\cdot\mathcal{A}^a \tag{11}$$

which is exactly the phase factor that appears in the Wilson line in eq. (6). This establishes the correspondence between the Wilson line in eq. (4) and the classical field in an abelian-like pure gauge as given by eq. (5).

In QCD, the generating functional is given by

$$Z_{\rm QCD}[J,\eta,\bar\eta] = \int [dQ][d\bar\psi][d\psi]\ \det(\frac{\delta \partial_\mu Q^{\mu a}}{\delta\omega^b})\ e^{i\int d^4x[-\frac{1}{4}{F^a}_{\mu\nu}^2[Q]-\frac{1}{2\alpha}(\partial_\mu Q^{\mu a})^2+\bar\psi D\!\!\!\!/[Q]\psi + J\cdot Q+\bar\eta\psi+\eta\bar\psi]} \tag{12}$$

Under the infinitesimal gauge transformation the quantum gluon field Q_μ^a transforms as

$$\delta Q_\mu^a = -g f^{abc}\omega^b Q_\mu^c + \partial_\mu\omega^a. \tag{13}$$

In the background field method of QCD the generating functional is given by [6, 7]

$$Z_{\rm background\ QCD}[A,J,\eta,\bar\eta] = \int [dQ][d\bar\psi][d\psi]\ \det(\frac{\delta G^a(Q)}{\delta\omega^b})$$
$$e^{i\int d^4x[-\frac{1}{4}{F^a}_{\mu\nu}^2[A+Q]-\frac{1}{2\alpha}(G^a(Q))^2+\bar\psi D\!\!\!\!/[A+Q]\psi + J\cdot Q+\bar\eta\psi+\eta\bar\psi]} \tag{14}$$

where A_μ^a is the background field. The gauge fixing term is given by

$$G^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c} = D_\mu[A] Q^{\mu a} \tag{15}$$

which depends on the background field A_μ^a. When the background field $A_\mu^a(x)$ is pure gauge in QCD given by

$$T^a A_\mu^a = \frac{1}{ig}(\partial_\mu U)U^{-1}, \qquad\qquad U = e^{igT^a\omega^a(x)} \tag{16}$$

we find from [9]

$$Z_{\rm QCD}[J,\eta,\bar\eta] \ = e^{i\int d^4x J\cdot A}\ \times\ Z_{\rm background\ QCD}[A,J,\eta,\bar\eta] - \int [dQ][d\bar\psi][d\psi]\ \det(\frac{\delta G_f^a(Q)}{\delta\omega^b})$$
$$e^{i\int d^4x[-\frac{1}{4}{F^a}_{\mu\nu}^2[Q]-\frac{1}{2\alpha}(G_f^a(Q))^2+\bar\psi D\!\!\!\!/[Q]\psi + J\cdot Q+\bar\eta\psi+\eta\bar\psi]}\ [i\int d^4x[J\cdot\delta Q + \bar\eta\delta\psi + \eta\delta\bar\psi + ...]], \tag{17}$$

where

$$G_f^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c} - \partial_\mu A^{\mu a} = D_\mu[A] Q^{\mu a} - \partial_\mu A^{\mu a}. \tag{18}$$

Changing the variable $Q \rightarrow Q - A$ in eq. (14) we find

$$Z_{\text{background QCD}}[A, J, \eta, \bar{\eta}] = e^{-i\int d^4x J\cdot A} \int [dQ][d\bar{\psi}][d\psi] \det(\frac{\delta G_f^a(Q)}{\delta \omega^b})$$
$$e^{i\int d^4x [-\frac{1}{4}{F^a}_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(G_f^a(Q))^2 + \bar{\psi} D\!\!\!\!/[Q]\psi + J\cdot Q + \eta\bar{\psi} + \bar{\eta}\psi]}. \tag{19}$$

The fermion fields and the corresponding sources transform as follows

$$\psi' = U\psi, \qquad \bar{\psi}' = \bar{\psi}U^{-1}, \qquad \eta' = U\eta, \qquad \bar{\eta}' = \bar{\eta}U^{-1}. \tag{20}$$

This gives

$$\delta(\eta\bar{\psi}) = \eta'\bar{\psi}' - \eta\bar{\psi} = 0 = \delta(\bar{\eta}\psi). \tag{21}$$

Using eq. (21) in (17) we find

$$Z_{\text{QCD}}[J, \eta, \bar{\eta}] \ = e^{i\int d^4x J\cdot A} \ \times \ Z_{\text{background QCD}}[A, J, \eta, \bar{\eta}] + \ \int [dQ][d\bar{\psi}][d\psi] \det(\frac{\delta G_f^a(Q)}{\delta \omega^b})$$
$$e^{i\int d^4x [-\frac{1}{4}{F^a}_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(G_f^a(Q))^2 + \bar{\psi} D\!\!\!\!/[Q]\psi + J\cdot Q + \bar{\eta}\psi + \eta\bar{\psi}]} \ [i\int d^4x[-J\cdot \delta Q + \psi\delta\bar{\eta} + \bar{\psi}\delta\eta + ...]]. \tag{22}$$

Hence from eqs. (22), (19) and (20) we finally obtain

$$Z_{\text{QCD}}[J, \eta, \bar{\eta}] \ = e^{i\int d^4x J\cdot A} \ \times \ Z_{\text{background QCD}}[A, J, \eta', \bar{\eta}'] - \ \int [dQ][d\bar{\psi}][d\psi] \det(\frac{\delta G_f^a(Q)}{\delta \omega^b})$$
$$e^{i\int d^4x [-\frac{1}{4}{F^a}_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(G_f^a(Q))^2 + \bar{\psi} D\!\!\!\!/[Q]\psi + J\cdot Q + \bar{\eta}\psi + \eta\bar{\psi}]} \ [i\int d^4x J\cdot \delta Q + ...]. \tag{23}$$

The correlation function in the presence of the background field $A_\mu^a(x)$ is given by

$$\frac{\delta}{\delta\eta(x_2)} \ \frac{\delta}{\delta\bar{\eta}(x_1)} \ Z_{\text{background QCD}}[A, J, \eta, \bar{\eta}]|_{J=\eta=\bar{\eta}=0} \quad = \quad <\bar{\psi}(x_2)\psi(x_1)>^A_{\text{background QCD}} . \tag{24}$$

The correlation function in QCD (without the background field) is given by

$$\frac{\delta}{\delta\eta(x_2)} \ \frac{\delta}{\delta\bar{\eta}(x_1)} \ Z_{\text{QCD}}[J, \eta, \bar{\eta}]|_{J=\eta=\bar{\eta}=0} \quad = \quad <\bar{\psi}(x_2)\psi(x_1)>^{A=0}_{\text{QCD}} . \tag{25}$$

Using eqs. (23), (24), (25) and (20) we find

$$<\bar{\psi}(x_2)U(x_2)U^{-1}(x_1)\psi(x_1)>^A_{\text{background QCD}} \ =<\bar{\psi}(x_2)\psi(x_1)>^{A=0}_{\text{QCD}} . \tag{26}$$

By taking appropriate color traces in eq. (26) and by using eqs. (11) and (6) we find

$$\text{Tr}_{\text{color}} \ <\bar{\psi}(x_2)\psi(x_1)>^A_{\text{background QCD}} \ = \ \text{Tr}_{\text{color}} \ [\mathcal{P}\ \exp[-ig\int_{-\infty}^0 d\lambda\ h\cdot \mathcal{A}^a(x_2^\mu + h^\mu\lambda)T^a]]$$
$$\times \ [<\bar{\psi}(x_2)\psi(x_1)>^{A=0}_{\text{QCD}}] \ \times \ [\bar{\mathcal{P}}\ \exp[ig\int_{-\infty}^0 d\lambda\ h\cdot \mathcal{A}^b(x_1^\nu + h^\nu\lambda)T^b]] \tag{27}$$

which proves the factorization theorem by using background field method of QCD. All the $\mathcal{A}_\mu^a(x)$ dependences (responsible for soft and collinear divergences) have been factored into the path ordered exponentials or Wilson lines. This concludes the proof of factorization theorem in high energy QCD through symmetry considerations at the lagrangian level.

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